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RESEARCH ARTICLE

Generation of strongly regular graphs from quaternary complex Hadamard matrices

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Abstract: A strongly regular graph with parameters (v,k,λ,μ) is a regular graph G with v vertices and k degree in which every two adjacent vertices have λ common neighbors and every two non-adjacent vertices have μ common neighbors. In this paper, we propose an algorithm which can be used to construct strongly regular graphs using quaternary complex Hadamard matrices. The order of the strongly regular graph generated by a quaternary complex Hadamard matrix of order n is n^2. The proposed algorithm has been illustrated by generating a strongly regular graph of order 4 using quaternary complex Hadamard matrix of order 2. Further, higher order strongly regular graphs were tested using Java program. This algorithm could be used to construct strongly regular graphs of order 2^{2n}; $n \in Z^{\wedge+}$.

Keywords: tensor product, simple graphs, k- regular graphs, strongly regular graphs, adjacency matrix, Latin squares, cyclic shifting method.

INTRODUCTION

Let n be even. A quaternary complex Hadamard matrix (Horadam, 2007) of order n is an $n \times n$ matrix H with entries from $\{\pm 1, \pm i\}$ such that $H(\overline{H})^T = nI_n$ where I_n is the identity matrix of order n. Following example gives the quaternary complex Hadamard matrix of order 2.

$$H = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$$

Hadamard matrices can be constructed using different techniques. One such technique is the tensor product or Kronecker product of Hadamard matrices.

Tensor product

If $A = \lfloor a_{ij} \rfloor$ is an $m \times m$ matrix and $B_1, B_2, \ldots, B_m$ are $n \times n$ matrices, with entries from a ring R, then the tensor product $A \otimes [B_1, B_2, \ldots, B_m]$ is the square matrix of order mn defined by (Again, 1985)

$$A \otimes [B_1, B_2, \ldots, B_m] = \begin{bmatrix} a_{11} B_1 & \ldots & a_{1m} B_1 \\ a_{21} B_2 & \ldots & a_{2m} B_2 \\ & & \\ a_{m1} B_m & \ldots & a_{mm} B_m \end{bmatrix}$$

Let H be a Hadamard matrix of order n. Then the partitioned matrix $\begin{bmatrix} H & H \\ H & -H \end{bmatrix}$ is a Hadamard matrix of order $2n$ obtained by Sylvester's construction (Wallis, 1975).

Further,

$$H_{2^k} = \begin{bmatrix} H_{2^{k-1}} & H_{2^{k-1}} \\ H_{2^{k-1}} & -H_{2^{k-1}} \end{bmatrix} = H_2 \otimes H_{2^{k-1}}$$

for $2 \le k \in \mathbb{N}$ where $\otimes$ denotes the kronecker product.

Simple graph

A graph G which has no loops and multi-edges is called a simple graph (West, 2000).A simple graph in which each vertex has the same degree k is named as *k*-regular graph (Diestel, 2005).

A regular graph $G = (V(G), E(G))$ is said to be strongly regular graph (Jayathilake *et al*, 2013) if there are integers λ and μ such that every two adjacent vertices have λ common neighbors and every two non-adjacent vertices have μ common neighbors which is defined by srg(v,k,λ,μ).The four parameters in an srg (v,k,λ,μ) are not independent and must obey the relation $(v-k-1)\mu = k(k-\lambda-1)$. For an example, Peterson graph has (10,3,0,1) parameters. (Figure 1)

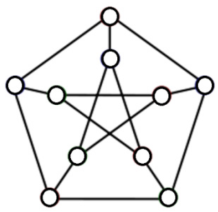

Figure 1: Peterson graph.

The adjacency matrix

The adjacency matrix of a simple graph G of order n is an $n \times n$ square matrix if its (i, j) entry is,

$$\begin{cases} 1 & ;\ \textit{if } i^{th} \textit{ vertex and } j^{th} \textit{ vertex are connected} \\ 0 & ;\ \textit{otherwise} \end{cases}$$

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Furthermore, the adjacency matrix A of a strongly regular graph satisfies $AJ=JA=kJ$ and $A^2+(\mu-\lambda)A+(\mu-k)I=\mu J$, where I denotes the identity matrix and J denotes the matrix whose all entries are 1 (Chartrand, 1985).

Latin square

A Latin square L of order n is an $n\times n$ array containing n different symbols such that each symbol occurs exactly once in each row and each column (Harris *et al*, 2000).

$$\begin{bmatrix}1&2&3&4\\3&4&1&2\\2&1&4&3\\4&3&2&1\end{bmatrix}$$

If $L=L^T$, then L is said to be Symmetric Latin square (van Lint, 2001). In our work Cyclic shifting method (Nishadi *et al*, 2017) is used to construct symmetric Latin squares.

$$\begin{bmatrix}1&2&3&4\\2&3&4&1\\3&4&1&2\\4&1&2&3\end{bmatrix}$$

In this paper, we have proposed an algorithm which can be used to generate strongly regular graphs from quaternary complex Hadamard matrices of order 2^n for $n\geq 1$.

METHODOLOGY

The method of formulating a srg(v,k,λ,μ) graph from quaternary complex Hadamard matrices is given as a recursive algorithm.

Steps of the proposed algorithm

Consider the quaternary complex Hadamard matrix of order 2 and label columns as column vectors c_1, c_2.

$$C_2=\begin{bmatrix}1&-i\\1&i\end{bmatrix} = \begin{bmatrix}c_1 & c_2\end{bmatrix}$$

Multiply each column vector by the transpose of its conjugate and label the resulting matrices as C_i. Applying it to the first column vector c_1 and second column vector c_2, the following block matrices can be obtained.

$$C_1=c_1\bar{c}_1^{\,T}=\begin{bmatrix}1\\1\end{bmatrix}\cdot\begin{bmatrix}1&1\end{bmatrix}=\begin{bmatrix}1&1\\1&1\end{bmatrix}$$

$$C_2=c_2\bar{c}_2^{\,T}=\begin{bmatrix}-i\\i\end{bmatrix}\cdot\begin{bmatrix}i&-i\end{bmatrix}=\begin{bmatrix}1&-1\\-1&1\end{bmatrix}$$

By using cyclic shifting method, symmetric Latin square of order 2, $\begin{bmatrix}C_1&C_2\\C_2&C_1\end{bmatrix}$ can be constructed and, replacing each element by the above constructed 2×2 matrices, and then replacing -1 by 1 and 1 by 0 the following adjacency matrix of strongly regular graph of order 4=2[2] can be obtained.

$$\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}$$

Construction of quaternary complex Hadamard matrix of order 4, C_4, from C_2 using Sylvester's construction is given below:

$$C_4=\begin{bmatrix}C_2&C_2\\C_2&-C_2\end{bmatrix}=\begin{bmatrix}1&-i&1&-i\\1&i&1&i\\1&-i&-1&i\\1&i&-1&-i\end{bmatrix}=\begin{bmatrix}c_1&c_2&c_3&c_4\end{bmatrix}$$

Then using the above construction $C_i^{'}$s can be formulated as follows:

$$C_i=c_i\bar{c}_i^{\,T}, \text{ where } i=1,2,3,4$$

$$C_1=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}.\begin{bmatrix}1&1&1&1\end{bmatrix}=\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}$$

$$C_2=\begin{bmatrix}-i\\i\\-i\\i\end{bmatrix}.\begin{bmatrix}i&-i&i&-i\end{bmatrix}=\begin{bmatrix}1&-1&1&-1\\-1&1&-1&1\\1&-1&1&-1\\-1&1&-1&1\end{bmatrix}$$

$$C_3=\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}.\begin{bmatrix}1&1&-1&-1\end{bmatrix}=\begin{bmatrix}1&1&-1&-1\\1&1&-1&-1\\-1&-1&1&1\\-1&-1&1&1\end{bmatrix}$$

$$C_4=\begin{bmatrix}-i\\i\\i\\-i\end{bmatrix}.\begin{bmatrix}i&-i&-i&i\end{bmatrix}=\begin{bmatrix}1&-1&-1&1\\-1&1&1&-1\\-1&1&1&-1\\1&-1&-1&1\end{bmatrix}$$

By using cyclic shifting method, the following Symmetric Latin square can be obtained.

$$\begin{bmatrix}C_1&C_2&C_3&C_4\\C_2&C_3&C_4&C_1\\C_3&C_4&C_1&C_2\\C_4&C_1&C_2&C_3\end{bmatrix}$$

The resulting adjacency matrix of strongly regular graph of order 16=4[2] as follows.

$$\begin{bmatrix}
0&0&0&0&0&1&0&1&0&0&1&1&0&1&1&0\\
0&0&0&0&1&0&1&0&0&0&1&1&1&0&0&1\\
0&0&0&0&0&1&0&1&1&1&0&0&1&0&0&1\\
0&0&0&0&1&0&1&0&1&1&0&0&0&1&1&0\\
0&1&0&1&0&0&1&1&0&1&1&0&0&0&0&0\\
1&0&1&0&0&0&1&1&1&0&0&1&0&0&0&0\\
0&1&0&1&1&1&0&0&1&0&0&1&0&0&0&0\\
1&0&1&0&1&1&0&0&0&1&1&0&0&0&0&0\\
0&0&1&1&0&1&1&0&0&0&0&0&0&1&0&1\\
0&0&1&1&1&0&0&1&0&0&0&0&1&0&1&0\\
1&1&0&0&1&0&0&1&0&0&0&0&0&1&0&1\\
1&1&0&0&0&1&1&0&0&0&0&0&1&0&1&0\\
0&1&1&0&0&0&0&0&0&1&0&1&0&0&1&1\\
1&0&0&1&0&0&0&0&1&0&1&0&0&0&1&1\\
1&0&0&1&0&0&0&0&0&1&0&1&1&1&0&0\\
0&1&1&0&0&0&0&0&1&0&1&0&1&1&0&0
\end{bmatrix}$$

We can obtain the strongly regular graphs by applying the same procedure for the quaternary complex Hadamard matrix of order 2^n for $n \geq 1$ obtained from Sylvester's construction.

RESULTS AND DISCUSSION

It can be seen that each block that are constructed by the operation $C_m = c_m \overline{c}_m{}^T$ for $m = 2,4,...$ are symmetric ($C_m = C_m{}^T$). Every undirected graph has symmetric property. Above method is used to construct strongly regular graph from the Quaternary complex Hadamard matrices of order $m = 2,4$. A Computer program has been developed by using proposed algorithm, to obtained Strong regular graphs (v,k,λ,μ) with large number of vertices. Java Programming and C+ language were used to write the above programme and the following figures were obtained.

Using the quaternary complex Hadamard matrix of order 2, we can obtain srg(4,1,0,0)(Figure 2). This graph contains 4 vertices with only one degree and every two adjacent vertices and two non-adjacent vertices have not any common neighbours.

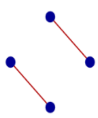

Figure 2: The strongly regular graph obtained from the quaternary complex Hadamard matrix C_2.

Also, srg (16,6,2,2)(Figure 3) can be obtained from the quaternary complex Hadamard matrix of order 4.This graph contains 16 vertices for which each vertex has the degree 6 and every two adjacent vertices have 2 common neighbours and every non-adjacent vertices have 2 common neighbours.

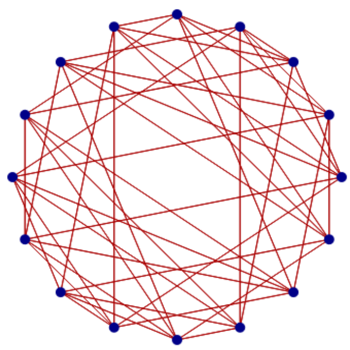

Figure 3: The strongly regular graph obtained from the quaternary complex Hadamard matrix C_4.

Using the quaternary complex Hadamard matrix of order 8, we can obtain srg (64,28,12,12)(Figure 4).

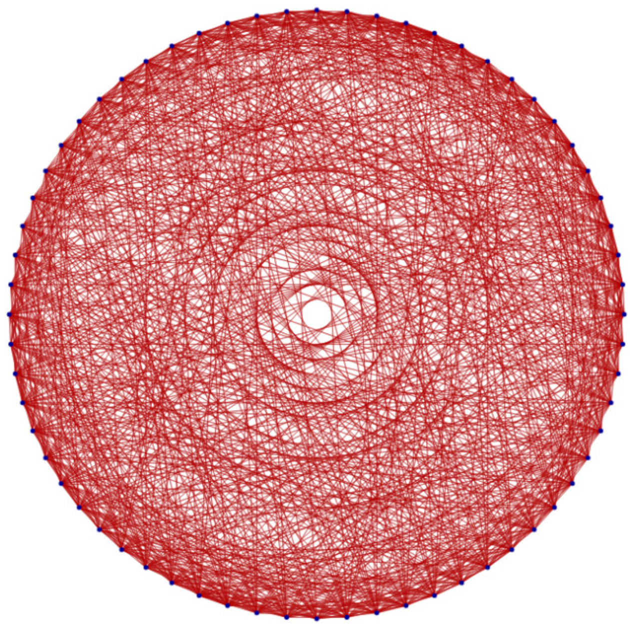

Figure 4: The strongly regular graph obtained from the quaternary complex Hadamard matrix C_8.

CONCLUSION

Regular graph of m^2 vertices has been constructed from quaternary Complex Hadamard matrix of order m where $m = 2^k$, $k \in \mathbb{N}$. Also, the adjacency matrix of the relevant graph can be obtained and it has symmetric property. Furthermore, the adjacency matrix A of a constructed regular graph satisfies $AJ=JA=kJ$ and $A^2 + (\mu-\lambda)A + (\mu-k)I = \mu J$, where I denotes the identity matrix and J denotes the matrix whose all entries are 1. Thus obtained graphs are strongly regular graphs with parameters $v = 4n^2, k = 2n^2 - n$ and $\lambda = \mu = n^2 - n$. Other observation is the strongly regular graphs obtained from the normalized Hadamard matrices and that of quaternary complex Hadamard matrices are equivalent.

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