Ceylon Journal of Science 47(3) 2018: 253-261
DOI: http://doi.org/10.4038/cjs.v47i3.7541

RESEARCH ARTICLE

Cubic Spline Solution of linear fourteenth order boundary value problems

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Received:11/05/2018; Accepted:07/07/2018

Abstract: As higher order differential equations have constantly been tiresome and problematic to resolve for the mathematicians and engineers so diverse numerical procedures were conceded out to acquire numerical estimates to such problems. In this paper an innovative numerical procedure is developed to estimate the fourteenth-order boundary value problems (BVPs) using Polynomial and Non-Polynomial Cubic spline. The procedures adopted in our work are based on cubic polynomial and non-polynomial spline method together with the decomposition procedure. In this paper polynomial and non-polynomial cubic splines along with the finite difference approximations will be used to squeeze the system of second order Boundary Value Problems in such a way that it will be converted into to a system consists of linear algebraic equations along with boundary conditions. These strategies will be operated on two problems to evidence the handiness of the technique by means of step size h = 1/5. The exactness of this method for detailed investigation is equated with the precise solution and conveyed through tables.

Keywords: Fourteenth order, Numerical Solution, Central finite difference approximations, Non-polynomial spline, System of linear algebraic equations, Polynomial Spline.

MATHEMATICS SUBJECT CLASSIFICATION (2010)

34K10. 34K28. 42A10. 65D05. 65D07.

Outline

A numerical procedure with polynomial and non-polynomial cubic splines have been established for procurement an estimated solution of a fourteenth order linear BVP, and this equation takes form

$$\begin{aligned}
&u^{(14)}(z) + a_1(z)\,u^{(13)}(z) + a_2(z)\,u^{(12)}(z) + \\
&a_3(z)\,u^{(11)}(z) + a_4(z)u^{(10)}(z) + a_5(z)\,u^{(9)}(z) + \\
&a_6(z)\,u^{(8)}(z) + a_7(z)\,u^{(7)}(z) + a_8(z)\,u^{(6)}(z) + \\
&a_9(z)\,u^{(5)}(z) + a_{10}(z)\,u^{(4)}(z) + a_{11}(z) \\
&u^{(3)}(z) + a_{12}(z)\,u^{(2)}(z) + a_{13}(z)\,u^{(1)}(z) + \\
&a_{14}(z)\,u(z) \;=\; f(z)\;;\; z \in [a,\, b] \qquad (1)
\end{aligned}$$

along with the boundary conditions:

$$u^{(2i)}(a) = j_i \;, \quad u^{(2i)}(b) = k_i \qquad (2)$$

where $(j_i,\ k_i\ ;\ i = 0, 1, \ldots, 6)$ are arbitrary static real constants, $f(z)$ and $(a_i(z)\ ;\ i = 1, 2, \ldots, 14)$ are continuous functions on the given interval [a, b].

Up-to-date studies in hydrodynamic and hydro magnetic stability have exposed the presence of a class of characteristic-value problems in differential equations of high order which have genuine mathematical concern. In (Akram and Nadeem, 2017) non-polynomial cubic spline procedure was used for the solution of ninth ODEs. The solution of the discussed above fourteen order linear differential using the numerical technique "fourth order Runge Kutta method" was described in (Chapra and Canale, 1998). In beam theory, the behavior equations and the obligatory boundary and continuity conditions for the whole arrangements are consequent explicitly, using five unknowns: the horizontal and vertical deflections of the upper and lower skins and the shear stress in the core. These equations are equivalent to one fourteenth order DE in terms of the unknowns as explained in (Frostig *et al*., 1992). In (Frostig and Thomsen, 2007) they clarified that the nonlinear governing equations for the radially symmetric circular sandwich plate case can be shown by a set of fourteen ODEs.

In (Hakeem *et al*., 2015), second order "linear klein-gordon" equation was solved by non-polynomial cubic spline. In (Justine and Sulaiman, 2017), solution of 4th order two point BVPs was found with non-polynomial spline. Non-Polynomial Spline Method for One Dimensional Nonlinear Benjamin-Bona-Mahony-Burgers Equation was used in (Kanth and Deepika, 2017). The authors developed non-polynomial quintic spline solution for the system of third order boundary-value problems in (Khan and Sultana, 2012). In (Khan *et al*., 2005), non-polynomial "quantic spline" solution of system of 3rd order boundary value problems and for sixth order BVPs was found. In (Pervaiz *et al*., 2014-a) and (Pervaiz and Ahmad, 2015), numerical solutions of sixth and twelve order BVPs were found by applying cubic non-polynomial spline method.

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In (Pervaiz *et al*., 2014-b), system of BVPs was solved by non-polynomial spline. In (Ramadan *et al*., 2008), a "septic non-polynomial spline" solution was used for sixth order BVPs.

Numerical solutions of eighth, tenth order BVPs and system of 4th order by cubic non-polynomial spline were found in (Siddiqi and Akram, 2007-c), (Siddiqi and Akram, 2007-b) and (Siddiqi and Akram, 2007-a). A non-polynomial spline method was established in (Taiwo and Ogunlaran, 2011) for linear 4th order BVPs. Non-polynomial splines approach to the solution of sixth-order boundary-value problems was established in (Tirmizi and Khan, 2008). In (Zarebnia *et al*., 2011), a non-polynomial spline technique was introduced for problems accruing in calculus of variations.

Here equation (1) is solved along accompanying boundary conditions (2) by reducing equation (1) to squeeze the system of second order BVPs in such a way that it will be converted into a system consists of linear algebraic equations along with boundary conditions of the following form:

$$r^{(2)}(z)+a_1(z)\,r^{(1)}(z)+a_2(z)\,r\ (z)+a_3(z)\,q^{(1)}(z)+ a_4(z)\,q(z)+a_5(z)p^{(1)}(z)+ a_6(z)\,p(z)+a_7(z)\,w^{(1)}(z)+a_8(z)\,w(z)+ a_9(z)\,v^{(1)}(z)+a_{10}(z)\,v\ (z)+a_{11}(z) y^{(1)}(z)+a_{12}(z)\,y(z)+a_{13}(z)\,u^{(1)}(z)+ a_{14}(z)\,u(z)\ =f(z)\,;\quad z\in[a,\,b] \tag{3}$$

$$\begin{aligned} u''(z) &= y(z)\\ y''(z) &= v(z)\\ v''(z) &= w(z)\\ w''(z) &= p(z)\\ p''(z) &= q(z)\\ q''(z) &= r(z) \end{aligned} \tag{4}$$

along with boundary conditions

$$\begin{aligned} u(a) &= j_0\,,\quad u(b)=k_0\\ y(a) &= j_1\,,\quad y(b)=k_1\\ v(a) &= j_2\,,\quad v(b)=k_2\\ w(a) &= j_3\,,\quad w(b)=k_3\\ p(a) &= j_4\,,\quad p(b)=k_4\\ q(a) &= j_5\,,\quad q(b)=k_5\\ r(a) &= j_6\,,\quad r(b)=k_6 \end{aligned} \tag{5}$$

Explanation of Non polynomial spline technique

In order to originate non-polynomial spline estimate S to (4) beside with the boundary conditions given in (5), we distribute the interval $[a,\,b]$ into n identical subintervals by means of grid points:

$$z_i = a + ih\,,\quad i=0\,,1\,,...,\,n$$

where $a=z_0,\ b=z_n,\ h=\dfrac{b-a}{n}$

and n is somewhat arbitrary positive integer.

Let $v(z)$ be the precise solution and v_i be an estimate to $v(z_i)$ attained by the non-polynomial cubic spline $S_i(z)$ through the points $(z_i,\,v_i)$ and $(z_{i+1},\,v_{i+1})$. It is obligatory for $S_i(z)$ to fulfill the interpolating conditions at z_i and z_{i+1}, the boundary conditions (5) and likewise the continuity of initial derivative at the mutual nodes $(z_i,\,v_i)$. For each segment $(z_i,\,z_{i+1})$ where $i=0,\,1,\,...,\,n-1$, the spline $S_i(z)$ takes the form

$$S_i(z)=s_i+g_i\,(z-z_i)+h_i\,sin\,\tau\,(z-z_i) + k_i\,cos\,\tau\,(z-z_i) \tag{6}$$

Where s_i, g_i, h_i, and k_i are constants and τ is an unrestricted parameter. A non polynomial function $S(z)$ of class $C^2[a,b]$ which interpolates $v(z)$ at the grid points z_i; $i=0\,,1\,,...,\,n$ be influenced by a parameter τ and reduces to a cubic spline $S(z)$ in $[a,b]$ as $\tau\to 0$.

For the derivation of coefficients of equation (6) in terms of $v_i,\ v_{i+1},\ N_i,\ N_{i+1}$, we first define

$$S_i(z_i)=v_i\,,\quad S_i(z_{i+1})=v_{i+1}\,,\quad S_i''(v_i)=N_i\,,\quad S_i''(v_{i+1})=N_{i+1} \tag{7}$$

By means of the simple algebraic manipulation we attain the consequent expressions for the coefficients of (6) as

$$s_i=v_i+\frac{N_i}{\tau^2},$$

$$g_i=\frac{v_{i+1}-v_i}{h}+\frac{N_{i+1}-N_i}{\tau\theta},$$

$$h_i=\frac{N_i\cos\theta-N_{i+1}}{\tau^2\sin\theta},\quad k_i=-\frac{N_i}{\tau^2}$$

Where $\theta=\tau h$, $\quad i=0,\,1,\,...,\,n-1$

Using the continuity condition of first derivative at the grid point $(z_i,\,v_i)$

i.e., $S'_{i-1}(z_i)=S'_i(z_i)$ is consistency relation for $i=0,\,1,\,...,\,n-1$

$$\alpha\,(N_{i+1}+N_{i-1})+2\beta\,N_i=\frac{1}{h^2}(v_{i-1}+v_{i+1}-2v_i) \tag{8}$$

Where $\alpha=\dfrac{1}{\theta\sin\theta}-\dfrac{1}{\theta^2}$, $\quad\beta=-\dfrac{1}{\theta^2}-\dfrac{\cos\theta}{\theta}$ and

$\theta=\tau h$

On similar lines we obtain relations for y, u, w, p, q and r respectively as;

$$\alpha\,(M_{i+1}+\,M_{i-1})+2\beta\,M_i\;=\;\frac{1}{h^2}(y_{i-1}+y_{i+1}-2y_i) \quad (9)$$

$$\alpha\,(L_{i+1}+\,L_{i-1})+2\beta\,L_i=\frac{1}{h^2}(u_{i-1}+u_{i+1}-2u_i) \quad (10)$$

$$\alpha\,(P_{i+1}+\,P_{i-1})+2\beta\,P_i=\frac{1}{h^2}(w_{i-1}+w_{i+1}-2w_i) \quad (11)$$

$$\alpha\,(Q_{i+1}+\,Q_{i-1})+2\beta\,Q_i=\frac{1}{h^2}(p_{i-1}+p_{i+1}-2p_i) \quad (12)$$

$$\alpha\,(R_{i+1}+\,R_{i-1})+2\beta\,R_i=\frac{1}{h^2}(q_{i-1}+q_{i+1}-2q_i) \quad (13)$$

$$\alpha\,(W_{i+1}+\,W_{i-1})+2\beta\,W_i\;=\;\frac{1}{h^2}(r_{i-1}+r_{i+1}-\,2r_i\,) \quad (14)$$

we have substituted

$$\begin{aligned} u'' &= L\\ y'' &= M\\ v'' &= N\\ w'' &= P\\ p'' &= Q\\ q'' &= R\\ r'' &= W \end{aligned} \quad (15)$$

The described method is fourth order convergent if $1-2\alpha-2\beta=0$ and $\alpha=\frac{1}{12}$ as described in (Taiwo and Ogunlaran, 2011).

Applications of non-polynomial spline

To demonstrate how the method is applied as urbanized in the preceding segment, we discretize (4) at the grid points (z_i, r_i), (z_i, u_i), (z_i, y_i), (z_i, v_i), (z_i, w_i), (z_i, p_i) *and* (z_i, q_i)

we have

$$\begin{aligned} &r_i^{(2)}(z_i)+a_1(z_i)\,r_i^{(1)}(z_i)+a_2(z_i)\,r_i\,(z_i)+\\ &a_3(z_i)\,q_i^{(1)}(z_i)+a_4(z_i)\,q_i\,(z_i)+\\ &a_5(z_i)\,p_i^{(1)}(z_i)+a_6(z_i)\,p_i\,(z_i)+a_7(z_i)w_i^{(1)}(z_i)+\\ &a_8(z_i)\,w_i\,(z_i)+a_9(z_i)\\ &v_i^{(1)}(z_i)+a_{10}(z_i)\,v_i\,(z_i)+a_{11}(z_i)\,y_i^{(1)}(z_i)+\\ &a_{12}(z_i)\,y_i\,(z_i)+a_{13}(z_i)\,u_i^{(1)}(z_i)\\ &a_{14}(z_i)\,u_i(z_i)\;=f_i(z_i) \end{aligned} \quad (16)$$

$$\begin{aligned} u_i'' &= y\,(z_i)\;=\;y_i\\ y_i'' &= v\,(z_i)\;=\;v_i\\ v_i'' &= w\,(z_i)\;=\;w_i\\ w_i'' &= p\,(z_i)\;=\;p_i\\ p_i'' &= q\,(z_i)\;=\;q_i\\ q_i'' &= r\,(z_i)\;=\;r_i \end{aligned} \quad (17)$$

and substituting

$$\begin{aligned} u_i'' &= L_i\\ y_i'' &= M_i\\ v_i'' &= N_i\\ w_i'' &= P_i\\ p_i'' &= Q_i\\ q_i'' &= R_i\\ r_i'' &= W_i \end{aligned} \quad (18)$$

and now rewriting, we get

$$\begin{aligned} W_i =\; &f_i-a_1(z_i)\,r_i^{(1)}-a_2(z_i)\,r_i\;-a_3(z_i)\,q_i^{(1)}-\\ &a_4(z_i)\,q_i\;-a_5(z_i)\,p_i^{(1)}-a_6(z_i)\,p_i\\ &-a_7(z_i)\,w_i^{(1)}-a_8(z_i)\,w_i\;-a_9(z_i)\,v_i^{(1)}-\\ &a_{10}(z_i)\,v_i\;-a_{11}(z_i)\,y_i^{(1)}-a_{12}(z_i)\,y_i\;-\\ &a_{13}(z_i)\,u_i^{(1)}-a_{14}(z_i)\,u_i \end{aligned} \quad (19)$$

$$\begin{aligned} L_i &= y_i\\ M_i &= v_i\\ N_i &= w_i\\ P_i &= p_i\\ Q_i &= q_i\\ R_i &= r_i \end{aligned} \quad (20)$$

From equation (19) and equation (20) we can write that

$$\begin{aligned} W_{i+1} =\; &f_{i+1}-a_1(z_{i+1})\,r_{i+1}^{(1)}-a_2(z_{i+1})\,r_{i+1}\;-\\ &a_3(z_{i+1})\,q_{i+1}^{(1)}-a_4(z_{i+1})\,q_{i+1}\;-a_5(z_{i+1})\\ &p_{i+1}^{(1)}-a_6(z_{i+1})\,p_{i+1}\;-a_7(z_{i+1})\,w_{i+1}^{(1)}-\\ &a_8(z_{i+1})\,w_{i+1}\;-a_9(z_{i+1})\,v_{i+1}^{(1)}-a_{10}(z_{i+1})\\ &v_{i+1}\;-a_{11}(z_{i+1})\,y_{i+1}^{(1)}-a_{12}(z_{i+1})\,y_{i+1}\;-\\ &a_{13}(z_{i+1})\,u_{i+1}^{(1)}-a_{14}(z_{i+1})\,u_{i+1} \end{aligned} \quad (21)$$

$$\begin{aligned} L_{i+1} &= y_{i+1}\\ M_{i+1} &= v_{i+1}\\ N_{i+1} &= w_{i+1}\\ P_{i+1} &= p_{i+1}\\ Q_{i+1} &= q_{i+1}\\ R_{i+1} &= r_{i+1} \end{aligned} \quad (22)$$

and

$$\begin{aligned} W_{i-1} =\; &f_{i-1}-a_1(z_{i-1})\,r_{i-1}^{(1)}-a_2(z_{i-1})\,r_{i-1}\;-\\ &a_3(z_{i-1})\,q_{i-1}^{(1)}-a_4(z_{i-1})\,q_{i-1}\;-a_5(z_{i-1})\\ &p_{i-1}^{(1)}-a_6(z_{i-1})\,p_{i-1}\;-a_7(z_{i-1})\,w_{i-1}^{(1)}-\\ &a_8(z_{i-1})\,w_{i-1}\;-a_9(z_{i-1})\,v_{i-1}^{(1)}-a_{10}(z_{i-1})\\ &v_{i-1}\;-a_{11}(z_{i-1})\,y_{i-1}^{(1)}-a_{12}(z_{i-1})\,y_{i-1}\;-\\ &a_{13}(z_{i-1})\,u_{i-1}^{(1)}-a_{14}(z_{i-1})\,u_{i-1} \end{aligned} \quad (23)$$

$$L_{i-1} = y_{i-1}$$
$$M_{i-1} = v_{i-1}$$
$$N_{i-1} = w_{i-1}$$
$$P_{i-1} = p_{i-1}$$
$$Q_{i-1} = q_{i-1}$$
$$R_{i-1} = r_{i-1} \qquad (24)$$

The following approximation of $O(h^2)$ for the first order derivative of u, y, v, w, p, q and r in equations (19), (21) and (23) can be used;

$$u_i' \cong \frac{u_{i+1} - u_{i-1}}{2h}, u_{i+1}' \cong \frac{3u_{i+1} - 4u_i + u_{i-1}}{2h}, u_{i-1}' \cong \frac{-u_{i+1} + 4u_i - 3u_{i-1}}{2h}, y_i' \cong \frac{y_{i+1} - y_{i-1}}{2h}$$
$$y_{i+1}' \cong \frac{3y_{i+1} - 4y_i + y_{i-1}}{2h}, y_{i-1}' \cong \frac{-y_{i+1} + 4y_i - 3y_{i-1}}{2h}, v_i' \cong \frac{v_{i+1} - v_{i-1}}{2h},$$
$$v_{i+1}' \cong \frac{3v_{i+1} - 4v_i + v_{i-1}}{2h}, v_{i-1}' \cong \frac{-v_{i+1} + 4v_i - 3v_{i-1}}{2h}, w_i' \cong \frac{w_{i+1} - w_{i-1}}{2h}$$
$$w_{i+1}' \cong \frac{3w_{i+1} - 4w_i + w_{i-1}}{2h}, w_{i-1}' \cong \frac{-w_{i+1} + 4w_i - 3w_{i-1}}{2h}, p_i' \cong \frac{p_{i+1} - p_{i-1}}{2h},$$
$$p_{i+1}' \cong \frac{3p_{i+1} - 4p_i + p_{i-1}}{2h}, p_{i-1}' \cong \frac{-p_{i+1} + 4p_i - 3p_{i-1}}{2h}, q_i' \cong \frac{q_{i+1} - q_{i-1}}{2h},$$
$$q_{i+1}' \cong \frac{3q_{i+1} - 4q_i + q_{i-1}}{2h}, q_{i-1}' \cong \frac{-q_{i+1} + 4q_i - 3q_{i-1}}{2h}, r_i' \cong \frac{r_{i+1} - r_{i-1}}{2h},$$
$$r_{i+1}' \cong \frac{3r_{i+1} - 4r_i + r_{i-1}}{2h}, r_{i-1}' \cong \frac{-r_{i+1} + 4r_i - 3r_{i-1}}{2h} \qquad (25)$$

using equations (19)-(25) in equations (8)-(14) we get

$$\alpha\, y_{i+1} + 2\beta\, y_i + \alpha\, y_{i-1} = \frac{1}{h^2}(u_{i+1} - 2u_i + u_{i-1})$$
$$\alpha\, v_{i+1} + 2\beta\, v_i + \alpha\, v_{i-1} = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1})$$
$$\alpha\, w_{i+1} + 2\beta\, w_i + \alpha\, w_{i-1} = \frac{1}{h^2}(v_{i+1} - 2v_i + v_{i-1})$$
$$\alpha\, p_{i+1} + 2\beta\, p_i + \alpha\, p_{i-1} = \frac{1}{h^2}(w_{i+1} - 2w_i + w_{i-1})$$
$$\alpha\, q_{i+1} + 2\beta\, q_i + \alpha\, q_{i-1} = \frac{1}{h^2}(p_{i+1} - 2p_i + p_{i-1})$$
$$\alpha\, r_{i+1} + 2\beta\, r_i + \alpha\, r_{i-1} = \frac{1}{h^2}(q_{i+1} - 2q_i + q_{i-1}) \qquad (26)$$

and

$$[\frac{2}{h^2} - 2\beta a_2(z_i) + \frac{2}{h}\alpha a_1(z_{i+1}) - \frac{2}{h}\alpha a_1(z_{i-1})]r_i - [\frac{1}{h^2} - \frac{1}{h}\beta a_1(z_i) + \frac{1}{2h}\alpha a_1(z_{i+1}) - \frac{3}{2h}\alpha$$
$$a_1(z_{i-1}) + \alpha a_2(z_{i-1})]r_{i-1} - [\frac{1}{h}\beta a_1(z_i) + \frac{3}{2h}\alpha a_1(z_{i+1}) + \frac{1}{h^2} - \alpha a_2(z_{i+1}) - \frac{1}{2h}\alpha a_1(z_{i-1})]$$
$$r_{i+1} + [\frac{1}{h}\beta a_3(z_i) + \frac{3}{2h}\alpha a_3(z_{i-1}) - \alpha a_4(z_{i-1}) - \frac{1}{2h}\alpha a_3(z_{i+1})]q_{i-1} + [\frac{2}{h}\alpha a_3(z_{i+1}) - 2\beta a_4(z_i)$$
$$-\frac{2}{h}\alpha a_3(z_{i-1})]q_i + [\frac{1}{2h}\alpha a_3(z_{i-1}) - \frac{3}{2h}\alpha a_3(z_{i+1}) - \alpha a_4(z_{i+1}) - \frac{1}{h}\beta a_3(z_i)]q_{i+1} + [\frac{1}{h}\beta a_5(z_i) +$$
$$\frac{3}{2h}\alpha a_5(z_{i-1}) - \alpha a_6(z_{i-1}) - \frac{1}{2h}\alpha a_5(z_{i+1})]p_{i-1} + [\frac{2}{h}\alpha a_5(z_{i+1}) - 2\beta a_6(z_i) - \frac{2}{h}\alpha a_5(z_{i-1})]p_i$$
$$+ [\frac{1}{2h}\alpha a_5(z_{i-1}) - \frac{3}{2h}\alpha a_5(z_{i+1}) - \alpha a_6(z_{i+1}) - \frac{1}{h}\beta a_5(z_i)]p_{i+1} + [\frac{1}{2h}\alpha a_7(z_{i-1}) - \frac{3}{2h}\alpha$$
$$a_7(z_{i+1}) - \alpha a_8(z_{i+1}) - \frac{1}{h}\beta a_7(z_i)]w_{i+1} + [\frac{2}{h}\alpha a_7(z_{i+1}) - 2\beta a_8(z_i) - \frac{2}{h}\alpha a_7(z_{i-1})]w_i + [\frac{1}{h}\beta$$

$$a_7(z_i)-\frac{1}{2h}\alpha a_7(z_{i+1})+\frac{3}{2h}\alpha a_7(z_{i-1})-\alpha a_8(z_{i-1})]w_{i-1}\ +\ [\frac{1}{h}\beta a_9(z_i)+\frac{1}{2h}\alpha a_9(z_{i-1})-\alpha a_{10}(z_{i+1})$$
$$-\frac{3}{2h}\alpha a_9(z_{i+1})]v_{i+1}\ +\ [\frac{1}{h}\beta a_9(z_i)-\frac{1}{2h}\alpha a_9(z_{i+1})+\alpha\frac{3}{2h}a_9(z_{i-1})-\alpha a_{10}(z_{i-1})]v_{i-1}\ +\ [-2\beta$$
$$a_{10}(z_i)-\frac{2}{h}\alpha a_9(z_{i-1})-\frac{2}{h}\alpha a_9(z_{i+1})]v_i\ +\ [\frac{3}{2h}\alpha a_{11}(z_{i-1})-\frac{1}{2h}\alpha a_{11}(z_{i+1})-\alpha a_{12}(z_{i-1})+\frac{1}{h}\beta$$
$$a_{11}(z_i)]y_{i-1}\ +\ [\frac{1}{2h}\alpha a_{11}(z_{i-1})-\alpha a_{12}(z_{i+1})-\frac{1}{h}\beta a_{11}(z_i)-\frac{3}{2h}\alpha a_{11}(z_{i+1})]y_{i+1}\ +\ [-2\beta a_{12}(z_i)\ -\frac{2}{h}$$
$$\alpha a_{11}(z_{i-1})+\frac{2}{h}\alpha a_{11}(z_{i+1})]y_i\ +\ [-\frac{2}{h}\alpha a_{13}(z_{i+1})-2\beta a_{14}(z_i)-\frac{2}{h}\alpha a_{13}(z_{i-1})]u_i\ +\ [-\alpha a_{14}(z_{i+1})-\frac{3}{2h}$$
$$\alpha a_{13}(z_{i+1})-\frac{1}{h}\beta a_{13}(z_i)+\frac{1}{2h}\alpha a_{13}(z_{i-1})]u_{i+1}\ +\ [\frac{1}{h}\beta a_{13}(z_i)-\frac{1}{2h}\alpha a_{13}(z_{i+1})+\alpha\frac{3}{2h}a_{13}(z_{i-1})$$
$$-\alpha a_{14}(z_{i-1})]u_{i-1}\ =\ -\ \alpha f_{i+1}\ -\ 2\beta f_i\ -\ \alpha f_{i-1} \qquad (27)$$

Equations (26) and (27) along the boundary conditions (5) provide a comprehensive system of $7\,(n\,+\,1)$ linear equations in $7\,(n\,+\,1)$ unknowns, which can be attained by relating modest numerical procedures.

Explanation of polynomial spline technique

In order to originate polynomial spline estimate S to (4) beside with the boundary conditions (5), we distribute the interval $[a,\ b]$ into n identical subintervals by means of grid points:

$z_i\ =\ a\ +\ ih\ ,\quad i=0\ ,\ 1\ ,\ ...,\ n$ where $a\ =\ z_0,\ b\ =\ z_n, h\ =\ \frac{b\ -\ a}{n}$
and n is somewhat arbitrary positive integer.

Then the spline $S_i(z)$ takes the form

$$S_i(z)=s_i\ +\ g_i\ \ (z\ -\ z_i)\ +\ h_i\ \ sin\,\tau\ \ (z-z_i)\ \ cos\,\tau\ \ (z-z_i) \qquad (28)$$

Where s_i , g_i , h_i , and k_i are constants and τ is an unrestricted parameter. Using the continuity condition of first derivative at the grid point $(z_i,\ v_i\ )$

i.e., $S'_{i\text{-}1}\ (z_i) = S'_i(z_i)$ is consistency relation for $i\ =\ 0,\ 1,\ ...,\ n-1$

$$N_{i+1}+N_{i-1}+4N_i=\frac{6}{h^2}(v_{i-1}+v_{i+1}-2v_i) \qquad (29)$$

On similar lines we obtain relations for y , u , w , p , q and r respectively as;

$$M_{i+1}+M_{i-1}+4M_i=\frac{6}{h^2}(y_{i-1}+y_{i+1}-2y_i) \qquad (30)$$

$$L_{i+1}+L_{i-1}+4L_i=\frac{6}{h^2}(u_{i-1}+u_{i+1}-2u_i) \qquad (31)$$

$$P_{i+1}+P_{i-1}+4P_i=\frac{6}{h^2}(w_{i-1}+w_{i+1}-2w_i) \qquad (32)$$

$$Q_{i+1}+Q_{i-1}+4Q_i=\frac{6}{h^2}(p_{i-1}+p_{i+1}-2p_i) \qquad (33)$$

$$R_{i+1}+R_{i-1}+4R_i=\frac{6}{h^2}(q_{i-1}+q_{i+1}-2q_i) \qquad (34)$$

$$W_{i+1}\ +\ W_{i-1}+\ 4W_i\ =\frac{6}{h^2}(\ r_{i-1}+r_{i+1}-\ 2r_i\ ) \qquad (35)$$

In a similar means to that for the non-polynomial cubic spline scheme, the polynomial cubic spline scheme was assembled as follows

$$[\frac{12}{h^2}-4a_2(z_i)+\frac{2}{h}a_1(z_{i+1})-\frac{2}{h}a_1(z_{i-1})]r_i-[\frac{6}{h^2}-\frac{2}{h}a_1(z_i)+\frac{1}{2h}a_1(z_{i+1})-\frac{3}{2h}$$
$$a_1(z_{i-1})+a_2(z_{i-1})]r_{i-1}-[\frac{2}{h}a_1(z_i)+\frac{3}{2h}a_1(z_{i+1})+\frac{6}{h^2}-a_2(z_{i+1})-\frac{1}{2h}a_1(z_{i-1})]$$
$$r_{i+1}+[\frac{2}{h}a_3(z_i)+\frac{3}{2h}a_3(z_{i-1})-a_4(z_{i-1})-\frac{1}{2h}a_3(z_{i+1})]q_{i-1}+[\frac{2}{h}a_3(z_{i+1})-4a_4(z_i)$$
$$-\frac{2}{h}a_3(z_{i-1})]q_i+[\frac{1}{2h}a_3(z_{i-1})-\frac{3}{2h}a_3(z_{i+1})-a_4(z_{i+1})-\frac{2}{h}a_3(z_i)]q_{i+1}+[\frac{2}{h}a_5(z_i)+$$
$$\frac{3}{2h}a_5(z_{i-1})-a_6(z_{i-1})-\frac{1}{2h}a_5(z_{i+1})]p_{i-1}+[\frac{2}{h}a_5(z_{i+1})-4a_6(z_i)-\frac{2}{h}a_5(z_{i-1})]p_i$$
$$+[\frac{1}{2h}a_5(z_{i-1})-\frac{3}{2h}a_5(z_{i+1})-a_6(z_{i+1})-\frac{2}{h}a_5(z_i)]p_{i+1}+[\frac{1}{2h}a_7(z_{i-1})-\frac{3}{2h}$$
$$a_7(z_{i+1})-a_8(z_{i+1})-\frac{2}{h}a_7(z_i)]w_{i+1}+[\frac{2}{h}a_7(z_{i+1})-4a_8(z_i)-\frac{2}{h}a_7(z_{i-1})]w_i+[\frac{2}{h}$$
$$a_7(z_i)-\frac{1}{2h}a_7(z_{i+1})+\frac{3}{2h}a_7(z_{i-1})-a_8(z_{i-1})]w_{i-1}+[\frac{2}{h}a_9(z_i)+\frac{1}{2h}a_9(z_{i-1})-$$
$$a_{10}(z_{i+1})-\frac{3}{2h}a_9(z_{i+1})]v_{i+1}+[\frac{2}{h}a_9(z_i)-\frac{1}{2h}a_9(z_{i+1})+\frac{3}{2h}a_9(z_{i-1})-a_{10}(z_{i-1})]v_{i-1}+$$
$$[-4a_{10}(z_i)-\frac{2}{h}a_9(z_{i-1})-\frac{2}{h}a_9(z_{i+1})]v_i+[\frac{3}{2h}a_{11}(z_{i-1})-\frac{1}{2h}a_{11}(z_{i+1})-a_{12}(z_{i-1})+\frac{2}{h}$$
$$a_{11}(z_i)]y_{i-1}+[\frac{1}{2h}a_{11}(z_{i-1})-a_{12}(z_{i+1})-\frac{2}{h}a_{11}(z_i)-\frac{3}{2h}a_{11}(z_{i+1})]y_{i+1}+[-4a_{12}(z_i)\ -$$
$$\frac{2}{h}a_{11}(z_{i-1})+\frac{2}{h}a_{11}(z_{i+1})]y_i+[-\frac{2}{h}a_{13}(z_{i+1})-4a_{14}(z_i)-\frac{2}{h}a_{13}(z_{i-1})]u_i+[-a_{14}(z_{i+1})-$$
$$\frac{3}{2h}a_{13}(z_{i+1})-\frac{2}{h}a_{13}(z_i)+\frac{1}{2h}a_{13}(z_{i-1})]u_{i+1}+[\frac{2}{h}a_{13}(z_i)-\frac{1}{2h}a_{13}(z_{i+1})+\frac{3}{2h}a_{13}(z_{i-1})$$
$$-a_{14}(z_{i-1})]u_{i-1}=-f_{i+1}-4f_i-f_{i-1} \qquad (36)$$

Numerical applications

In this segment we assess the conclusion of the technique established by resolving two BVPs with step size $h=\frac{1}{5}$.
The numerical outcomes and graphical comparisons of exact solution and our solution are offered in tables and figures.

Example 1

$$\frac{d^{14}u\ (z)}{dz^{14}}\ =\ e^{-z}u\ (z); \qquad 0\le z\le 1$$

subject to

$$u^{(2i)}\ (0)\ =\ 1\ ,\quad u^{(2i)}\ (1)\ =\ e$$

where $(\ i\ =\ 0\ ,1\ ,\ ...,\ 6\ )$ with exact soluti
on $u\ (z)\ =\ e^{z}$. Numerical results are shown in the Table 1.

Table 1: Comparison of Cubic Polynomial and Cubic Non-Polynomial of problem 1.

z	Exact solution	Cubic Non-Polynomial Solution	Cubic Polynomial Solution	Cubic Non-Polynomial absolute error	Cubic Polynomial absolute error	Cubic Non-Polynomial reletive error	Cubic Polynomial reletive error
0.2	1.2214027581	1.2214020778	1.2214020778	6.80E-07	3.71E-04	5.57E-07	3.04E-04
0.4	1.4918246976	1.4918236132	1.4918236132	1.08E-06	5.92E-04	7.24E-07	3.97E-04
0.6	1.8221188003	1.8221176295	1.8221176295	1.17E-06	6.35E-04	6.42E-07	3.48E-04
0.8	2.2255409284	2.2255400739	2.2255400739	8.54E-07	4.59E-04	3.84E-07	2.06E-04

Graphical comparison of absolute errors of both splines is plotted in Graph 1.

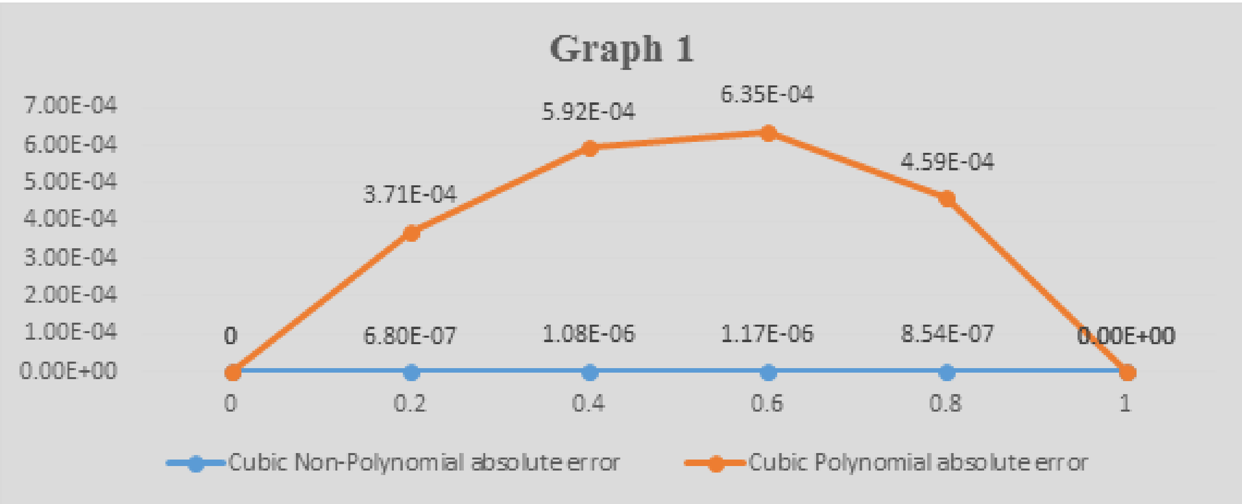

Figure 1: Graphical Comparison of Cubic Polynomial and Cubic Non-Polynomial of problem 1.

The values of $u^{(2i)}\ (z)$ where $(i = 0, 1, ..., 6)$ are given in Table 2.

Table 2: Comparison of all derivatives of Cubic Polynomial and Cubic Non-Polynomial of problem 1.

z	Cubic Non-Polynomial $u^{(2i)}\ (z)$ where $(i = 0, 1, ..., 6)$	Cubic Polynomial $u^{(2i)}\ (z)$ where $(i = 0, 1, ..., 6)$
0.2	1.2214020778	1.2214020778
0.4	1.4918236132	1.4918236132
0.6	1.8221176295	1.8221176295
0.8	2.2255400739	2.2255400739

Example 2

$$\frac{d^{14}u\ (z)}{dz^{14}} = Cos\ (z) - Sin\ (z) \qquad 0 \leq z \leq 1$$

subject to

$$u^{(2i)}\ (0) = 1 \quad , \ u^{(2i)}\ (1) = Cos\ (1)\ + \ Sin\ (1)$$
$$u^{(2j)}\ (0) = -1\ , \ u^{(2j)}\ (1) = -Cos\ (1)\ - \ Sin\ (1)$$

where i = 0, 4, 8, 12 and j = 2, 6, 10 with exact solution

$u\ (z)\ =\ Cos\ (z)\ +\ Sin\ (z)$. Numerical results are shown in the Table 3.

Table 3: Comparison of Cubic Polynomial and Cubic Non-Polynomial of problem 2.

z	Exact solution	Cubic Non-Polynomial Solution	Cubic Polynomial Solution	Cubic Non-Polynomial absolute error	Cubic Polynomial absolute error	Cubic Non-Polynomial reletive error	Cubic Polynomial reletive error
0.2	1.1787359086	1.1787365267	1.1783589367	6.18E-07	3.77E-04	5.24E-07	3.20E-04
0.4	1.3104793363	1.3104802833	1.3098957142	9.47E-07	5.84E-04	7.23E-07	4.46E-04
0.6	1.3899780883	1.3899790569	1.3893837364	9.68E-07	5.94E-04	6.96E-07	4.27E-04
0.8	1.4140628002	1.4140634612	1.4136645389	6.60E-07	3.98E-04	4.67E-07	2.81E-04

Graphical comparison of absolute errors of both splines is plotted in Graph 2.

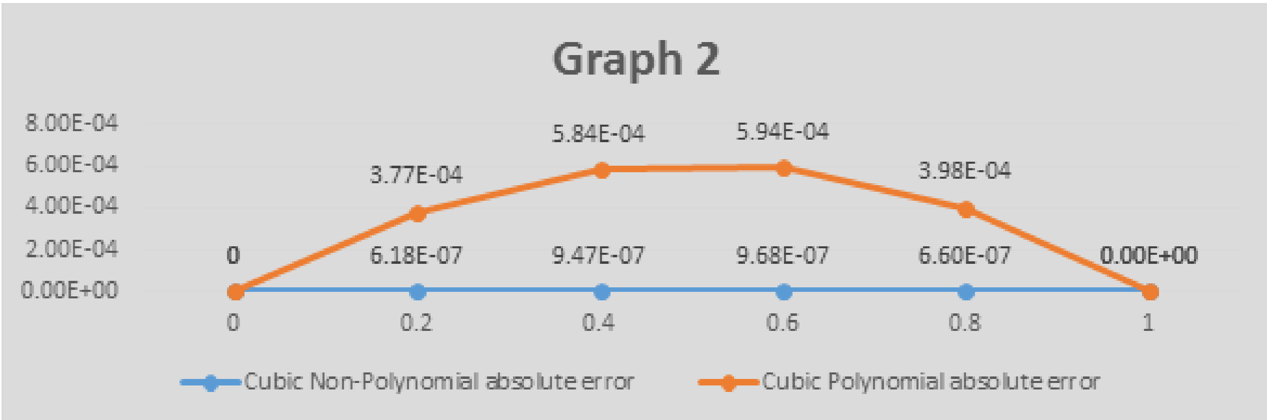

Figure 2: Comparison of Cubic Polynomial and Cubic Non-Polynomial.

The values of $u^{(2i)}\ (z)$ where $(i = 0, 1, ..., 6)$ are given in Table 4.

Table 4: Comparison all derivatives of Cubic Polynomial and Cubic Non-Polynomial of problem 2.

z	Cubic Non-Polynomial $u^{(2i)}\ (z)$ where $j = 2, 6, 10$	Cubic Polynomial $u^{(2i)}\ (z)$ where $j = 2, 6, 10$
0.2	-1.178735276	-1.1783575895
0.4	-1.310478260	-1.3098935343
0.6	-1.389977034	-1.3893815565
0.8	-1.414062211	-1.4136631917
z	Cubic Non-Polynomial $u^{(2i)}\ (z)$ where $i = 0, 4, 8, 12$	Cubic Polynomial $u^{(2i)}\ (z)$ where $i = 0, 4, 8, 12$
0.2	1.1787365267	1.1783589367
0.4	1.3104802833	1.3098957141
0.6	1.3899790569	1.3893837363
0.8	1.4140634612	1.4136645389

CONCLUSIONS

The fourteenth order BVPs have unusual numerical solution in literature, so it is significant to discover dependable numerical procedure for explaining these problems. Estimated solutions of the fourteenth order BVPs have been acquired by means of the Polynomial and Non-Polynomial Cubic Spline scheme. The technique is applied on two test problems and the outcomes attained are appropriately precise up to 7 decimal places as shown in tables and graphs which shows the legitimacy of the established process. The computations associated with the examples discussed above were performed by using MATLAB R2015a.

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