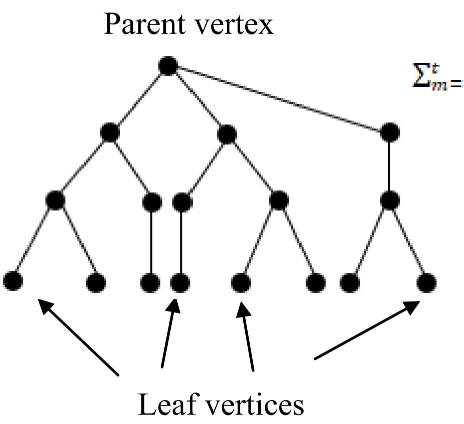
RESEARCH ARTICLE

Characterization of E-super vertex magic labelling of odd order binary trees

S.K.O.D. Samarathunge, A.M.C.U.M. Athapattu and A.A.I. Perera

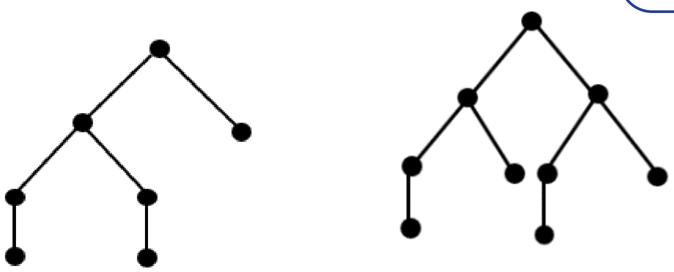
Lemma



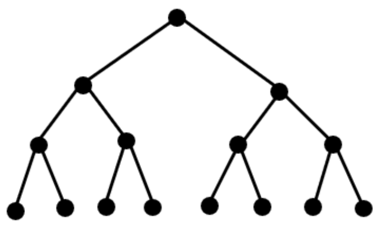
$\sum_{m=1}^{t} \lambda(e_{00\,m}) \geq k-(p+q)$ where $t = deg(u_{00})$.

$\lambda(e_{n-1\,j\,m}) \geq k-(p+q)$

There exists E-super vertex magic labelling for odd order binary trees.



Theorem



E-super vertex magic labelling does not exist for the perfect binary trees

Highlights

- Presented lemma can be used to find E-super vertex magic labelling for the odd order trees.
- It is possible to find E-super vertex magic labelling with at least two vertices of degree 2.
- Perfect binary tree T($2^{n+1}-1, 2^{n+1}-2$) is not a E-super vertex magic when $n \geq 2$.
- For any vertex u, the $\sum_{v \in N(u)} \lambda(uv)$ is distinct for any binary tree.

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Characterization of E-super vertex magic labelling of odd order binary trees

S.K.O.D. Samarathunge[1], A.M.C.U.M. Athapattu[2,*] and A.A.I. Perera[2]

[1]*David and Judi Proctor Department of Mathematics, Dodge Family College of Arts and Sciences, University of Oklahoma, Norman, USA*
[2]*Department of Mathematics, Faculty of Science, University of Peradeniya, Peradeniya, Sri Lanka*

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Abstract: In the field of Graph Theory, a tree is a connected acyclic undirected graph. Any tree with odd vertices is conjectured to be an E-super vertex magic. In this paper, we verified that the E-super vertex magic labelling does not exist for the perfect binary trees and extended the work to identify the possibility of finding E-super vertex magic labelling for odd order binary trees.

Keywords: *E*-super vertex magic labelling; Super vertex magic labelling; Tree graph.

INTRODUCTION

Graph Theory, a branch of mathematics, concerns with networks of points connected by lines. Graph labelling is one of the key branches in Graph Theory, which has its origins traced back to the mid-1960s. It has been an exciting area of research during the last few decades. Graph labelling has many applications in coding theory, x-ray crystallography, radar, astronomy, circuit design, communication network addressing, and database management. More detailed discussion regarding graph labelling techniques and its applications can be found in (Gallian, 2018).

A labelling of a graph is a function with domain as some set of graph elements like the set of vertices, the set of edges or both the set of edges and vertices, and whose ranges is a set of positive integers. Further, various restrictions can be placed on the function (Wallis, 2001).

Let $G = (V, E)$ be a finite simple graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges with the cardinalities of p and q, respectively. An Edge magic labelling of a G is an injective function λ from $V(G) \cup E(G)$ onto the integers $\{1, 2, \ldots, p + q\}$ with the property that, for a given edge (uv), $\lambda(u) + \lambda(uv) + \lambda(v) = k$. This labelling concept was originated by −(Sedláček, 1963). Afterword, antimagic graphs was introduced by Hartsfield and Ringel in 1990. A graph is called antimagic if its edges can be labelled with $1, 2, \ldots, q$ without repetition such that the sum of the vertex labels pairwise distinct. In particularly, sum of the labels of the edges incident to each vertex are distinct.

Later, various labelling techniques related to magic labelling were introduced. MacDougall *et al*.,(2002) introduced the notion of vertex magic total labelling and super vertex magic total labelling around 2000. A vertex magic total labelling is a bijection map λ from $V(G) \cup E(G)$ to the set $\{1, 2, \ldots, p + q\}$ with the property that for any vertex u, $\lambda(u) + \sum_{v \in N(u)} \lambda(uv) = k$, where k is a constant and the set $N(u)$ denotes the vertices adjacent to vertex u. Moreover, they call a vertex magic total labelling is super if $\lambda(V(G)) = \{1, 2, \ldots, p\}$. In their work, they have discussed the properties of these labellings, and constructed labellings for families of graphs such as cycles, paths, complete graphs of odd order and the complete bipartite graph (MacDougall *et al*., 2002). Swaminathan and Jeyanthi (2003) introduced a concept with the name super vertex magic labelling, but with different notion. They call a vertex magic total labelling is super, if $\lambda(E(G)) = \{1, 2, \ldots, q\}$. To avoid the misleading, Marimuthu and Balakrishnan (2012) named it as an *E*-super vertex magic (total) labelling. They studied some basic properties of such labelling. There are numerous literatures based on E-super vertex magic labelling (also called super vertex magic total labelling or strong vertex magic total labelling), see, Gray, 2006; Gray, 2007; Gray and MacDougall, 2009; Gray and MacDougall, 2012; Wang and Zhang, 2013 and Wang and Zhang, 2014.

In this work, we have considered *E*-super vertex magic labelling. Our work is motivated by the open problem; every tree is an E-super vertex magic (Marimuthu *et al*., 2015). We studied possibilities for the existence of E-super vertex magic labelling in some of the binary trees with odd order.

MATERIALS AND METHODS

In this section, we consider the *E*-super vertex magic labelling for binary trees, followed by related theorems from literature. An E-super vertex-magic labeling of a graph with p vertices and q edges is a one-to-one map taking the vertices and edges onto the set $\{1, 2, \ldots, p + q\}$ with the property that the sum of the label on a vertex and the labels on its incident edges is a constant which is independent of the choice of vertex.

Swaminathan and Jeyanthi (2003) proved that a path graph with *n* vertices is *E*-super vertex magic if and only if *n* is odd and $n \geq 3$. They also proved that if a non- trivial graph *G* is E-super vertex magic, then the magic constant *k* is,

**Corresponding Author's Email:* chathurikaa@sci.pdn.ac.lk
https://orcid.org/0000-0002-8012-5502

$$k = q + \frac{p+1}{2} + \frac{q(q+1)}{p}.$$

However, Marimuthu and Balakrishnan (2012) proved that if p is even, then every tree T is not E-super vertex magic.

In this paper, we try to characterize trees in terms of E-super vertex magic labelling. In particularly, we have considered the connected graph without any cycles. Now, we define the following notations to identify the edges and vertices of trees. Let us order from left to right at any level and denote the vertices in the level i of a tree as $u_{i\,1}, u_{i\,2}, u_{i\,3}, \cdots$. The notation u_{ij}, i denotes the level of the vertex and the j denotes the position of the vertex from the left to right ordering. Let us denote the edges that connect the level i and level $i+1$ in the left to right ordering. Edges incident with u_{ij} can be labeled as $e_{i\,j\,1}, e_{i\,j\,2}, e_{i\,j\,3}, \cdots$. That is, the notation $e_{ij\,m}$, i and j denote the subscript of the vertex u_{ij}, and m denotes the position of the edge from the left to right ordering within the considered vertex where $1 \le m < \deg(u_{ij})$.

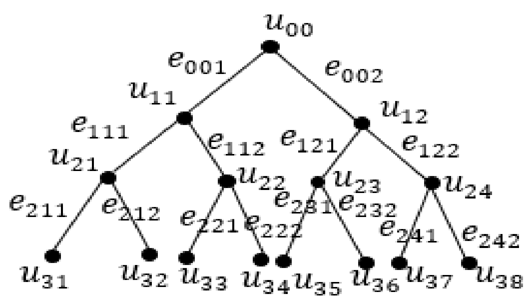


Figure 1: Labelled Tree.

For a better understanding, we can illustrate this using Figure 1. In the figure, we have used the above notations to identify vertices of tree with $p = 15$ vertices and $q = 14$ edges. Further, these notations will be used throughout the paper.

Note that, a tree with p vertices have $p-1$ edges.

The following lemma has been proved to obtain our main result.

Lemma: For a tree with odd order,

i. labellings of edges which are adjacent to leaf vertex should be greater than or equal to $k-(p+q)$.

i.e., for a binary tree with the height n,

$\lambda(e_{n-1\,j\,m}) \ge k - (p+q)$

where $j \in \mathbb{N}$ and $1 \le m < \deg(u_{n-1\,j})$.

ii. parent edge labellings should be greater than or equal to $k-(p+q)$.

i.e.,

$$\sum_{m=1}^{t} \lambda(e_{00\,m}) \ge k - (p+q)$$

where $t = deg(u_{00})$.

Proof: i. Let $p = |V(G)|$ and $q = |E(G)|$.

Define a total labelling;
$\lambda: V(G) \cup E(G) \to \{1,2,\ldots,p+q\}$

with the property that, for any vertex u_{ij},

$$\lambda(u_{ij}) + \sum_{m=1}^{t} \lambda(e_{i\,j\,m}) = k$$

where $t = deg(u_{ij})$.

By the definition of E- super vertex magic labelling, we have following inequalities. For edge labelling,

$$1 \le \lambda(e_{i\,j\,m}) \le q \longrightarrow (1).$$

For vertex labelling,

$$q+1 \le \lambda(u_{ij}) \le p+q \longrightarrow (2).$$

<u>Claim:</u> $k > p+q$

We know,

$$k = q + \frac{p+1}{2} + \frac{q(q+1)}{p}.$$

By the properties of binary trees $p = q+1$, we can write

$$k = \frac{5q+2}{2} = \frac{3q}{2} + q + 1 = \frac{1}{2}q + (p+q).$$

Thus, $k > p+q$.

We know that $k - \lambda(e_{n-1\,j\,m}) = \lambda(u_{n\,l})$.

Here, $u_{n\,l}$ is the vertex which is adjacent to $e_{n-1\,j\,m}$.

Here,

$$l = \sum_{r=1}^{j-1} \{deg(u_{n-1\,r}) - 1\} + m.$$

By (2), $q+1 \le k - \lambda(e_{n-1\,j\,m}) \le p+q$.

That is, $k - \lambda(e_{n-1\,j\,m}) \le p+q$.

Therefore, $\lambda(e_{n-1\,j\,m}) \ge k - (p+q)$.

i.e., $k-(p+q) \le \lambda(e_{n-1\,j\,m}) \le q$.

i. The proof of this part can be directly obtained by using the definition of E-super vertex magic labelling and the inequality (2) in the proof of part (i).

$$\sum_{m=1}^{t} \lambda(e_{00\,m}) \geq k-(p+q)$$

where $t = deg(u_{00})$.

For our research, binary trees and perfect binary trees have been used. A binary tree can be defined as a tree with every non-leaf having exactly two children and not necessary to use left to right ordering for the children (Foulds, 1992). In a complete binary tree, every level except possibly the last level, is completely filled, and all nodes in the last level are as far left as possible. It can have nodes between 1 and 2^n at the last level n. A binary tree is symmetric if the left subtree via the vertical line through the root is the mirror reflection of the right subtree.

RESULTS AND DISCUSSION

Perfect Binary Tree

A perfect binary tree is a binary tree in which all interior nodes have two children, and all leaves have the same depth or the same level. Therefore, the perfect binary tree can be denoted as T($2^{n+1}-1$, $2^{n+1}-2$). Here, $2^{n+1}-1$ and $2^{n+1}-2$ stand for the number of vertices and the number of edges, respectively, and n is the height of the tree. It is not possible to label these types of graphs as the number of edges in the highest level is equal to the number of integers that can be used to label the highest level. When we consider the same example (Figure 1), we have 8 edges in the highest level which is equal to the number of integers that can be used to label this level. i.e., by Lemma (i), possible integers for the labelling of the highest level are 7,8,9,10,11,12,13,14.

Theorem:

Perfect binary tree T($2^{n+1}-1$, $2^{n+1}-2$) is not a *E*-super vertex magic when $n \geq 2$.

Proof:

Let q be the number of edges in the tree. It is known that the number of edges in the highest level is 2^n.
Define a total labelling;
$\lambda: V(G) \cup E(G) \rightarrow \{1,2,\ldots,p+q\}$

with the property that for any vertex u_{ij},

$$\lambda(u_{ij}) + \sum_{m=1}^{t} \lambda(e_{i\,j\,m}) = k$$

where $t = deg(u_{ij})$.

Note that for E-super vertex magic labelling (Marimuthu and Balakrishnan, 2012),

$$\lambda(u_{n-1\,j}) \geq q+1.$$

Here, u_{ij} is the j^{th} vertex from left to right in the i^{th} level. Edges $e_{n-1\,j\,m}$ can be labelled according to the part (i) in the Lemma,

$$\lambda(e_{n-1\,j\,m}) \geq k-(p+q).$$

In particular, there are $q-[k-(p+q+1)]$ $q-[k-(p+q+1)]$ integers that can be used to label the edges in the highest level.

According to part (ii) in the Lemma,

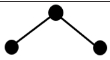
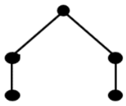
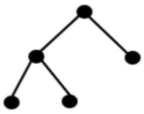
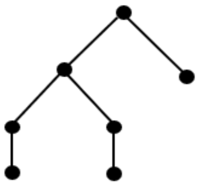
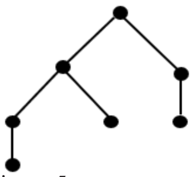
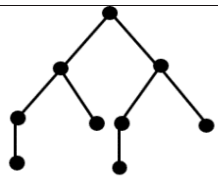
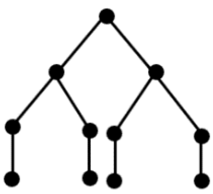
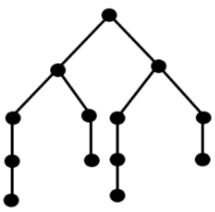
$$\lambda(e_{001}) + \lambda(e_{002}) \geq k-(p+q).$$

We know that number of edges in the highest level is equal to the number of integers that can be used to label the highest level.

Hence, there are no integers for the labelling of e_{001} and e_{002} as $\lambda(e_{001}) + \lambda(e_{002})$ implies that $\lambda(u_{00}) = \lambda(u_{n\,j})$ for some $j \leq 2^n$.

In addition to these, by the observation, we can say that for any vertex u, $\sum_{v \in N(u)} \lambda(uv)$ is distinct for any binary tree where the set $N(u)$ denotes the vertices adjacent to vertex u.

Table 1: E-super vertex magic labelling for odd order binary trees.

p	***q***	***k***	**Graph**	**Magic Labelling**
				Edges
3	2	6	 Figure 1	$\lambda(e_{001}) = 1$, $\lambda(e_{002}) = 2$,
5	4	11	 Figure 2	$\lambda(e_{001}) = 4$, $\lambda(e_{002}) = 1$, $\lambda(e_{111}) = 2$, $\lambda(e_{121}) = 3$.
			 Figure 3	$\lambda(e_{001}) = 1$, $\lambda(e_{002}) = 4$, $\lambda(e_{111}) = 3$, $\lambda(e_{112}) = 2$.
7	6	16	 Figure 4	$\lambda(e_{001}) = 2$, $\lambda(e_{002}) = 3$, $\lambda(e_{111}) = 5$, $\lambda(e_{112}) = 1$, $\lambda(e_{211}) = 4$, $\lambda(e_{221}) = 6$.
			 Figure 5	$\lambda(e_{001}) = 1$, $\lambda(e_{002}) = 2$, $\lambda(e_{111}) = 3$, $\lambda(e_{112}) = 5$, $\lambda(e_{121}) = 6$, $\lambda(e_{211}) = 4$.
9	8	21	 Figure 6	$\lambda(e_{001}) = 3$, $\lambda(e_{121}) = 2$, $\lambda(e_{002}) = 6$, $\lambda(e_{122}) = 4$, $\lambda(e_{111}) = 1$, $\lambda(e_{211}) = 5$, $\lambda(e_{112}) = 7$, $\lambda(e_{221}) = 8$.
11	10	26	 Figure 7	$\lambda(e_{001}) = 2$, $\lambda(e_{122}) = 1$, $\lambda(e_{002}) = 3$, $\lambda(e_{211}) = 7$, $\lambda(e_{111}) = 5$, $\lambda(e_{221}) = 6$, $\lambda(e_{112}) = 8$, $\lambda(e_{231}) = 9$, $\lambda(e_{121}) = 4$, $\lambda(e_{241}) = 10$.
13	12	31	 Figure 8	$\lambda(e_{001}) = 2$, $\lambda(e_{211}) = 5$, $\lambda(e_{002}) = 4$, $\lambda(e_{221}) = 11$, $\lambda(e_{111}) = 8$, $\lambda(e_{231}) = 6$ $\lambda(e_{112}) = 7$, $\lambda(e_{241}) = 12$, $\lambda(e_{121}) = 1$, $\lambda(e_{311}) = 9$, $\lambda(e_{122}) = 3$, $\lambda(e_{321}) = 10$.

CONCLUSIONS

In this study, we have proved a lemma which can be used to find E-super vertex magic labelling for binary trees. According to this lemma, labelling of edges which adjacent to leaf vertex should be greater than or equal to $k-(p+q)$, and sum of the parent edge labellings should be greater than or equal to $k-(p+q)$. Based on this result, it has been proved that E-super vertex magic labelling does not exist for the perfect binary trees with

$p=2^{n+1}-1, q=2^{n+1}-2; n \geq 2$.

Further, in Table 1, we have illustrated some examples for E-super vertex magic labelling of odd order binary trees. Here, we presernted the result for a single magic constant. The future studies may be focused on developing an algorithm to figure out different magic constants, and finding possibilities of the existence of E-super vertex magic labelling for symmetric binary trees. In particularly, it is an open problem whether, symmetric binary trees with 2^n-2 edges in the highest level is E-super vertex magic or not.

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DECLARATION OF CONFLICT OF INTEREST

There are no conflicting interests.

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